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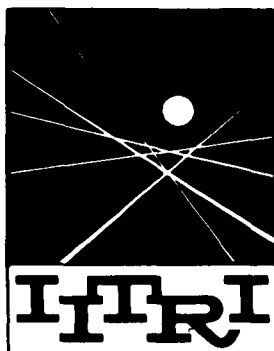
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ANALYSIS OF FRACTURE PROBABILITIES
IN NON-UNIFORMLY STRESSED
BRITTLE MATERIALS ,

(10) by

N. A. Weil and I. M. Daniel.

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ANALYSIS OF FRACTURE PROBABILITIES IN NON-UNIFORMLY STRESSED BRITTLE MATERIALS

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ABSTRACT

This paper describes the results of theoretical studies on the effect of non-uniform stress fields limited to those encountered in prismatic beams under bending, upon the fracture of brittle materials. Derivations were carried out to determine the risk of rupture of bending specimens subjected to a symmetrical four-point load of arbitrary spacing, the symmetric three-point loading and pure bending forming limiting cases of this more general loading. The analysis was based on materials obeying the Weibull distribution function with assumptions for either volumetric or surface flaw dispersion conditions. Comparisons are presented between the predicted strengths of bending and tensile specimens.

An analytical method for the determination of the three Weibull parameters from a pure bending test is proposed. This method, based on the "best fit" of a theoretical curve to the experimental data, was successfully applied to experimental results on Columbia Resin, a brittle amorphous polymer.

ANALYSIS OF FRACTURE PROBABILITIES IN NON-UNIFORMLY STRESSED BRITTLE MATERIALS

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I.   INTRODUCTION

In practical applications ceramic parts are almost invariably subjected to loading conditions resulting in   non-uniform internal stresses. In fact, because of the purely elastic nature of these substances, it is difficult to impose uniform stresses upon them even under the most carefully controlled conditions, so that most laboratory tests also employ one or another form of non-uniform stress condition.

In recent years an increased awareness has evolved that the fracture strength of ceramic substances can be satisfactorily represented only as a statistical quantity, and suitable approaches were developed to extend this statistical treatment to bodies subjected to non-uniform stresses. However, all of these theories assume that the cumulative fracture probability for the entire body is determined by summing the probability of fracture of its infinitesimal component elements, each subjected to a uniform state of stress.   The existence of a stress gradient, and the shear stresses that inevitably accompany its presence, are not assumed to contribute to failure by themselves.

To date there had been no attempt made to confirm the validity of this assumption.   An investigation was, therefore, undertaken to examine the influence of the non-uniformity of stress upon the probability of fracture, and to determine whether the existence of a stress gradient, per se, had a demonstrable effect upon the fracture strength.   This investigation was carried out in two complementary parts: an analytical and an experimental program.

This paper presents the results of theoretical studies concerning the influence of non-uniform stress fields upon the fracture characteristics of brittle materials.   The complementary experimental program was completed since this paper was submitted and results were presented recently[1]*.

The most widely accepted statistical theory of fracture is based on

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* Numbers in parentheses refer to References.

the Weibull distribution function[2, 3]. The theory uses two basic criteria of fracture, size and normal stress, and postulates that failure in an isotropic, homogeneous material is fully described by three material parameters: the "zero strength", the "flaw density exponent" and a scale parameter. Within the validity of these assumptions, the theory is capable of describing failure for any type of stress distribution, uniform or non-uniform, uniaxial or polyaxial. For these reasons, the Weibull theory was selected as the basis of the analytical work described here.

Analysis of fracture probabilities in the presence of stress gradient have been conducted by Weibull himself and more recently by V. Weiss[4, 5]. Although non-linear gradients were investigated the simplifying assumption of a zero value for the "zero strength" was made.

The non-uniform stress-field chosen for the present analysis was that of the simple beam subjected to four-point loading. This stress field is one of the simplest non-uniform stress fields that can be studied; in fact, for pure bending a single parameter, the stress gradient, is sufficient to describe the stress distribution. Also, this specimen shape and loading condition lends itself excellently to carefully controlled tests, and was used throughout the experimental work on this program.

Basic to any experimental work studying the effect of a given parameter is the need for obtaining a completely reliable fundamental statistical distribution of strengths for the material in question. Customarily, such is done either by a trial-and-error graphical method originally suggested by Weibull, or by analytical approaches devised by others[6, 7, 8, 9]. However, the graphical method cannot be freed from errors attributable to subjective judgments, and the analytical treatments are mostly suitable for a large number of purely tensile specimens.

Therefore, following the theoretical derivations for the effect of non-uniform stress fields, an analytical method is presented here for defining the "best fit" of a theoretical Weibull curve to a set of experimentally obtained data. The appropriate parameters are obtained by a minimization process of the sum of the mean squares. The procedure so developed is then demonstrated by applying it to a series of bending tests conducted on Columbia Resin (CR-39) specimens.

II.   WEIBULL THEORY

The Weibull theory uses two basic criteria of failure, size and normal tensile stress. For a uniaxial stress field in a homogeneous isotropic material, governed by volumetric flaw distribution, the probability of fracture at a given stress σ is given by

$$S = \begin{cases} 1 - \exp\left[-\int_V \left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m dV\right] = 1 - e^{-B}, & \sigma \geqslant \sigma_u \\ 0 & , \sigma < \sigma_u \end{cases} \tag{1}$$

where

$$B = \int_V \left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m dV \tag{2}$$

is the "risk of rupture", and

σ_u = zero probability strength (location parameter)

m = flaw density exponent (shape parameter)

σ_o = scale parameter

The last three parameters are associated with the material and are independent of size.

The mean failure stress is given by

$$\sigma_m = \sigma_u + \int_{\sigma_u}^{\infty} e^{-B} d\sigma \tag{3}$$

and the variance by

$$a^2 = \int_{\sigma_u}^{\infty} e^{-B} d(\sigma^2) + \sigma_u^2 - \sigma_m^2 \tag{4}$$

From Eq. 1 it is seen that the theory does not make any special allowance for the non-uniformity of stress distribution. Each infinitesimal element of a specimen is considered to be under uniform tensile stress, and the risk of rupture for the whole specimen is obtained by integrating the risk of rupture of each infinitesimal element over the volume of the speciment. The stress gradient does not enter as an independent parameter, and all the non-uniformity effects seem to be accounted for by the risk of rupture. The question of whether the stress gradient has an independent effect on the fracture stress is equivalent to the question of whether the Weibull theory is sufficient to predict failure for non-uniform stress fields.

In a material governed by surface flaw distribution the risk of rupture is given by

$$B = \int_A \left(\frac{\sigma - \sigma_u}{\sigma_o} \right)^m dA \tag{5}$$

In order to establish the dependence of the risk of rupture on the dimensions of a specimen and the type of loading, the risk of rupture was calculated for a general case of a prismatic beam under four-point loading. Derivations were made for both cases of volumetric and surface flaw distribution.

III.   **RISK OF RUPTURE**

A.   **Material Governed by Volumetric Flaw Distribution**

The distribution of tensile stresses in the beam shown in Fig. 1 is

$$\left. \begin{array}{l} \sigma = \dfrac{2 k \sigma_b}{hL} \; yx \;\; \text{for } 0 \leqslant x \leqslant \dfrac{L}{k} \;\; \text{and } (1 - \dfrac{1}{k}) L \leqslant x \leqslant L \\[4mm] \sigma = \dfrac{2 \sigma_b}{h} \; y \;\; \text{for } \dfrac{L}{k} \leqslant x \leqslant (1 - \dfrac{1}{k}) L \end{array} \right\} \tag{6}$$
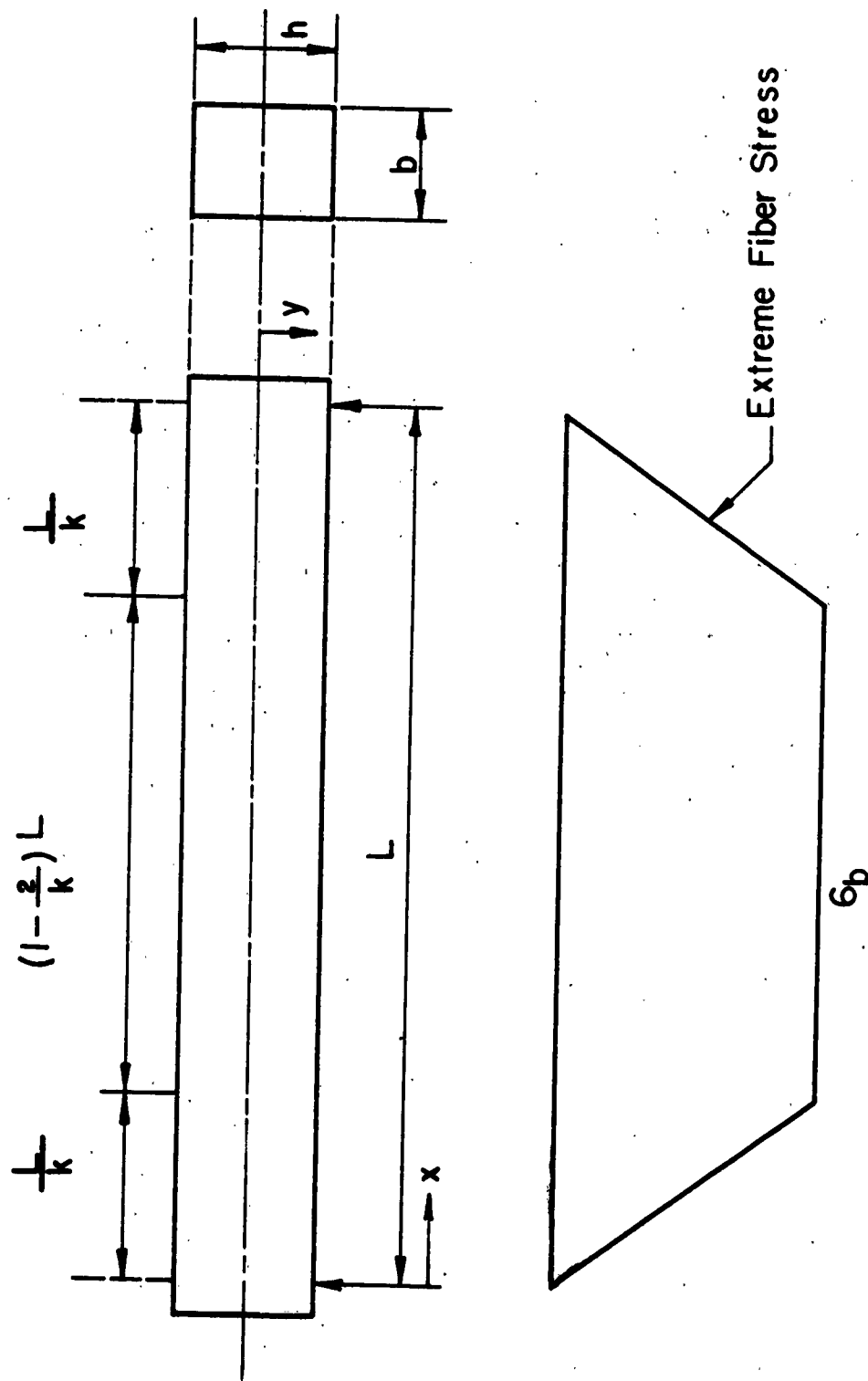
Fig. 1 PRISMATIC BEAM UNDER 4-POINT LOADING AND DISTRIBUTION OF EXTREME FIBER STRESS

The risk of rupture is then composed of two parts

$$B_b = B_b' + B_b''$$ (7)

where B_b' corresponds to the central portion of the specimen subjected to uniform bending and B_b'' refers to the outer portions. It can be shown that

$$B_b' = \frac{V\left(1 - \frac{2}{k}\right)}{2(m+1)}\left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m$$ (8)

and

$$B_b'' = \frac{V}{2(m+1)}\left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \frac{2}{k}\sum +$$

$$+ \frac{V}{k(m+1)\sigma_b\sigma_o^m}\left(-\sigma_u\right)^{[m]+1} I_\alpha$$ (9)

where

$$m = [m] + \alpha, \quad \text{with } [m] \text{ the largest integer less than or}$$
equal to m,

$$I_\alpha = \int_{y_u}^{h/2} \frac{1}{y}\left(\frac{2\sigma_b}{h}\, y - \sigma_u\right)^\alpha dy$$ (10)

$$y_u = \frac{h}{2}\, \frac{\sigma_u}{\sigma_b}$$

$$\sum = \sum_{r=0}^{[m]} \frac{1}{m+1-r} \left(1 - \frac{\sigma_b}{\sigma_u}\right)^{-r} \tag{11}$$

Using Eq. 7 we obtain

$$B_b = \frac{V}{2(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \left[1 - \frac{2}{k} + \frac{2}{k}\sum\right] +$$

$$+ \frac{V}{k(m+1)\sigma_b \sigma_o^{\ m}} (-\sigma_u)^{[m]+1} I_\alpha \tag{12}$$

If m is an integer, i.e. if $m = [m]$, then $\alpha = 0$ and Eq. 12 reduces to

$$B_b = \frac{V}{2(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \left[1 - \frac{2}{k} + \frac{2}{k}\sum\right] +$$

$$+ \frac{V}{k(m+1)\sigma_b \sigma_o^{\ m}} \left(-\sigma_u\right)^{m+1} \ell n \frac{h}{2y_u} \tag{13}$$

Specific forms of this formula can be obtained by assigning values to the parameter k.

a.   <u>Pure Bending ($k = \infty$)</u>

$$B_b = \frac{V}{2(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \tag{14}$$

b.   **Quadrant-Point Loading (k = 4)**

$$B_b = \frac{V}{4(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b}\right) \left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \left(1 + \sum\right)$$

$$+ \frac{V}{4(m+1)\,\sigma_b\,\sigma_o^{\ m}} \ (-\sigma_u)^{m+1} \ \ell n \ \frac{h}{2y_u}$$

(15)

c.   **Third-Point Loading (k = 3)**

$$B_b = \frac{V}{6(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b}\right) \left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \left(1 + 2\sum\right)$$

$$+ \frac{V}{3(m+1)\,\sigma_b\,\sigma_o^{\ m}} \left(-\sigma_u\right)^{m+1} \ \ell n \ \frac{h}{2y_u}$$

(16)

d.   **Three-Point Loading (Center-Point Loading; k = 2)**

$$B_b = \frac{V}{2(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b}\right) \left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \sum \ +$$

$$+ \frac{V}{2(m+1)\,\sigma_b\sigma_o^{\ m}} \left(-\sigma_u\right)^{m+1} \ \ell n \ \frac{h}{2y_u}$$

(17)

When $\sigma_u = 0$, both Eqs. 12 and 13 reduce to

$$B_b = \frac{V}{2(m+1)} \left(\frac{\sigma_b}{\sigma_o}\right)^m \ \frac{k(m+1) - 2m}{k(m+1)}$$

(18)

- 8 -

Again, we list below some typical cases of interest for specific values of k.

a.   Pure Bending (k = ∞)

$$B_b = \frac{V}{2(m+1)} \left(\frac{\sigma_b}{\sigma_o} \right)^m \tag{19}$$

b.   Quadrant-Point Loading (k = 4)

$$B_b = \frac{V}{4(m+1)} \left(\frac{\sigma_b}{\sigma_o} \right)^m \frac{m+2}{m+1} \tag{20}$$

c.   Third-Point Loading (k = 3)

$$B_b = \frac{V}{6(m+1)} \left(\frac{\sigma_b}{\sigma_o} \right)^m \frac{m+3}{m+1} \tag{21}$$

d.   Three-Point Loading (Center-Point Loading; k = 2)

$$B_b = \frac{V}{2(m+1)} \left(\frac{\sigma_b}{\sigma_o} \right)^m \frac{1}{m+1} \tag{22}$$

The variation of the risk of rupture with the spacing of concentrated loads on the beam is described by the factor

$$K = \frac{k(m+1) - 2m}{k(m+1)} \tag{23}$$

in Eq. 18. The relationship among the values of B_b given in Eqs. 19 through 22 is illustrated in Fig. 2 by plotting the factor K against the parameter m.
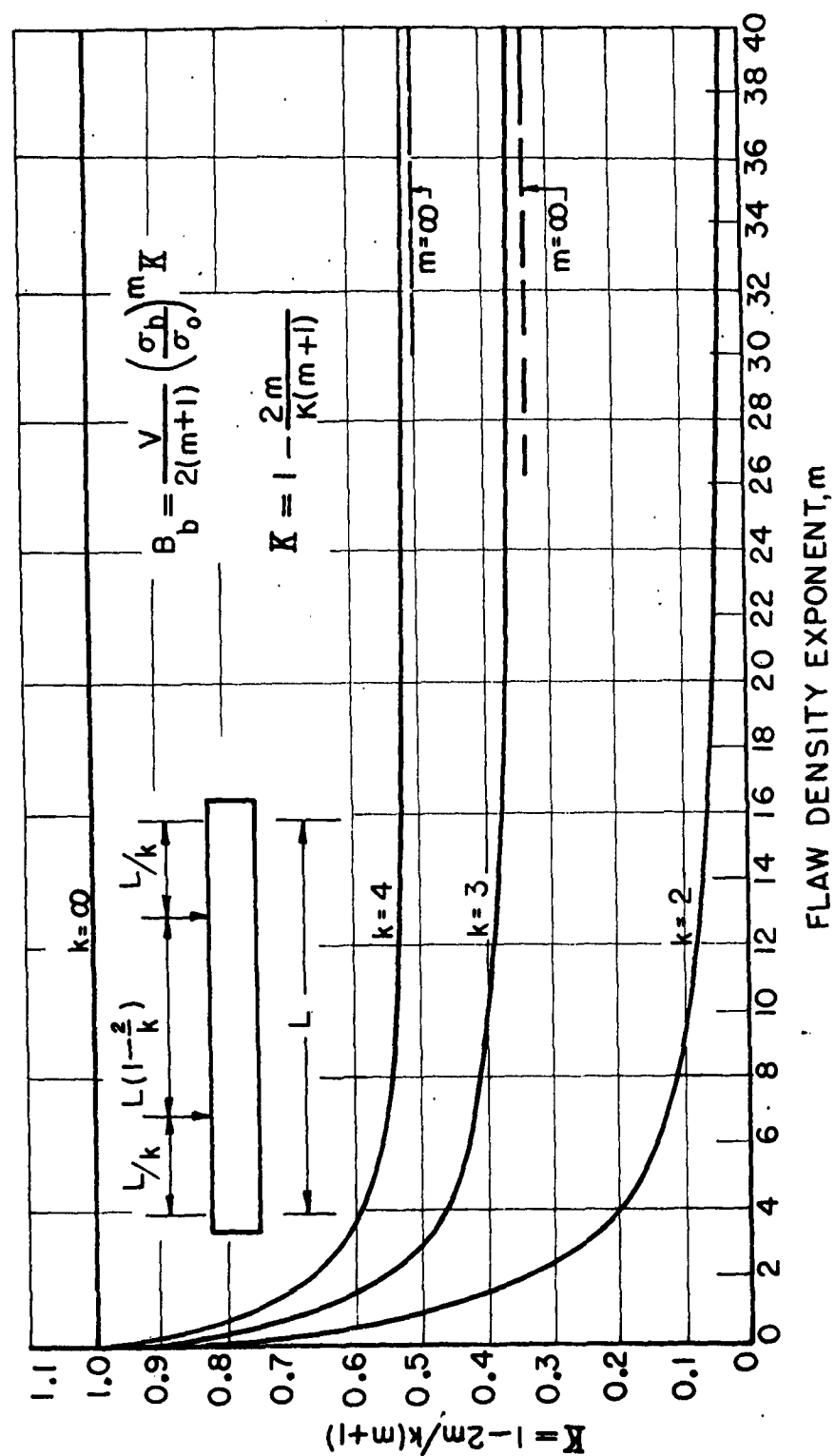
Fig. 2 LOADING FACTOR K VERSUS FLAW DENSITY EXPONENT m FOR MATERIAL GOVERNED BY VOLUMETRIC FLAW DISTRIBUTION

B.   Material Governed by Surface Flaw Distribution

For the stress distribution defined by Eq. 6 we have

$$B_b = \frac{L}{m+1}\left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m\left[\left(1 - \frac{2}{k}\right)h + \frac{2h}{k}\sum + \frac{2b}{k}\right]$$

$$+ \left(1 - \frac{2}{k}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m Lb + \frac{2hL}{k(m+1)\sigma_b\sigma_o{}^m}\left(-\sigma_u\right)^{[m]+1}I_\alpha \qquad (24)$$

If m is an integer, Eq. 23 reduces to

$$B_b = \frac{L}{m+1}\left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m\left[\left(1 - \frac{2}{k}\right)h + \frac{2h}{k}\sum + \frac{2b}{k}\right] +$$

$$\qquad (25)$$

$$+ \frac{2hL}{k(m+1)\sigma_b\sigma_o{}^m}\left(-\sigma_u\right)^{m+1}\ell n\,\frac{h}{2y_u} + \left(1 - \frac{2}{k}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m Lb$$

For the specific cases mentioned before we have

a.   Pure Bending (k = ∞)

$$B_b = L\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m\left[\frac{h}{m+1}\left(1 - \frac{\sigma_u}{\sigma_b}\right) + b\right] \qquad (26)$$

b.   Quadrant-Point Loading (k = 4)

$$B_b = \frac{L}{m+1}\left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m\left[\frac{h}{2} + \frac{h}{2}\sum + \frac{b}{2}\right]$$

$$+ \frac{2hL}{4(m+1)\sigma_b\sigma_o{}^m}\left(-\sigma_u\right)^{m+1}\ell n\,\frac{h}{2y_u} + \frac{1}{2}\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m Lb \qquad (27)$$

c.   Third-Point Loading (k = 3)

$$B_b = \frac{L}{m+1} \left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \left(\frac{h}{3} + \frac{2h}{3}\sum + \frac{2b}{3}\right) +$$

$$+ \frac{1}{3}\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m Lb + \frac{2hL}{3(m+1)\sigma_b\sigma_o^m}\left(-\sigma_u\right)^{m+1} \ell n \frac{h}{2y_u}$$

(28)

d.   Three-Point Loading (Center-Point Loading; k = 2)

$$B_b = \frac{L}{m+1}\left(1 - \frac{\sigma_u}{\sigma_b}\right)\left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m\left(h\sum + b\right) +$$

$$+ \frac{2hL}{2(m+1)\sigma_b\sigma_o^m}\left(-\sigma_u\right)^{m+1} \ell n \frac{h}{2y_u}$$

(29)

When $\sigma_u = 0$ both Eqs. 24 and 25 reduce to

$$B_b = L\left(\frac{\sigma_b}{\sigma_o}\right)^m\left(\frac{h}{m+1} + b\right)\frac{k(m+1) - 2m}{k(m+1)}$$

(30)

For the specific values of k considered above Eq. 30 yields:

a.   Pure Bending (k = ∞)

$$B_b = L\left(\frac{\sigma_b}{\sigma_o}\right)^m\left(\frac{h}{m+1} + b\right)$$

(31)

b.   Quadrant-Point Loading (k = 4)

$$B_b = L\left(\frac{\sigma_b}{\sigma_o}\right)^m\left(\frac{h}{m+1} + b\right)\frac{m+2}{2(m+1)}$$

(32)

c. **Third-Point Loading (k = 3)**

$$B_b = L \left(\frac{\sigma_b}{\sigma_o} \right)^m \left(\frac{h}{m+1} + b \right) \frac{m+3}{3(m+1)} \tag{33}$$

d. **Three-Point Loading (Center-Point Loading: k = 2)**

$$B_b = L \left(\frac{\sigma_b}{\sigma_o} \right)^m \left(\frac{h}{m+1} + b \right) \frac{1}{m+1} \tag{34}$$

Equation 30 contains the same loading factor K as Eq. 18. In the case of surface-distributed flaws the loading factor can be redefined and extended to include the effect of width to depth ratio of the beam. Thus, Eq. 30 can be rewritten as

$$B_b = A \left(\frac{\sigma_b}{\sigma_o} \right)^m D \tag{35}$$

where

A = the surface area of the beam and

$$D = \frac{1}{2(1+\lambda)} \left(\frac{1}{m+1} + \lambda \right) \frac{k(m+1) - 2m}{k(m+1)} \tag{36}$$

with

$$\lambda = \frac{b}{h}$$

The relationship among the values B_b given in Eqs. 31 through 34 is illustrated in Figs. 3 to 7 by plotting the combined loading-shape factor D versus parameter m.

IV.   RELATIONSHIP BETWEEN BENDING AND TENSILE STRENGTHS

For a material governed by volumetric flaw distribution the risk of rupture in a tensile specimen is

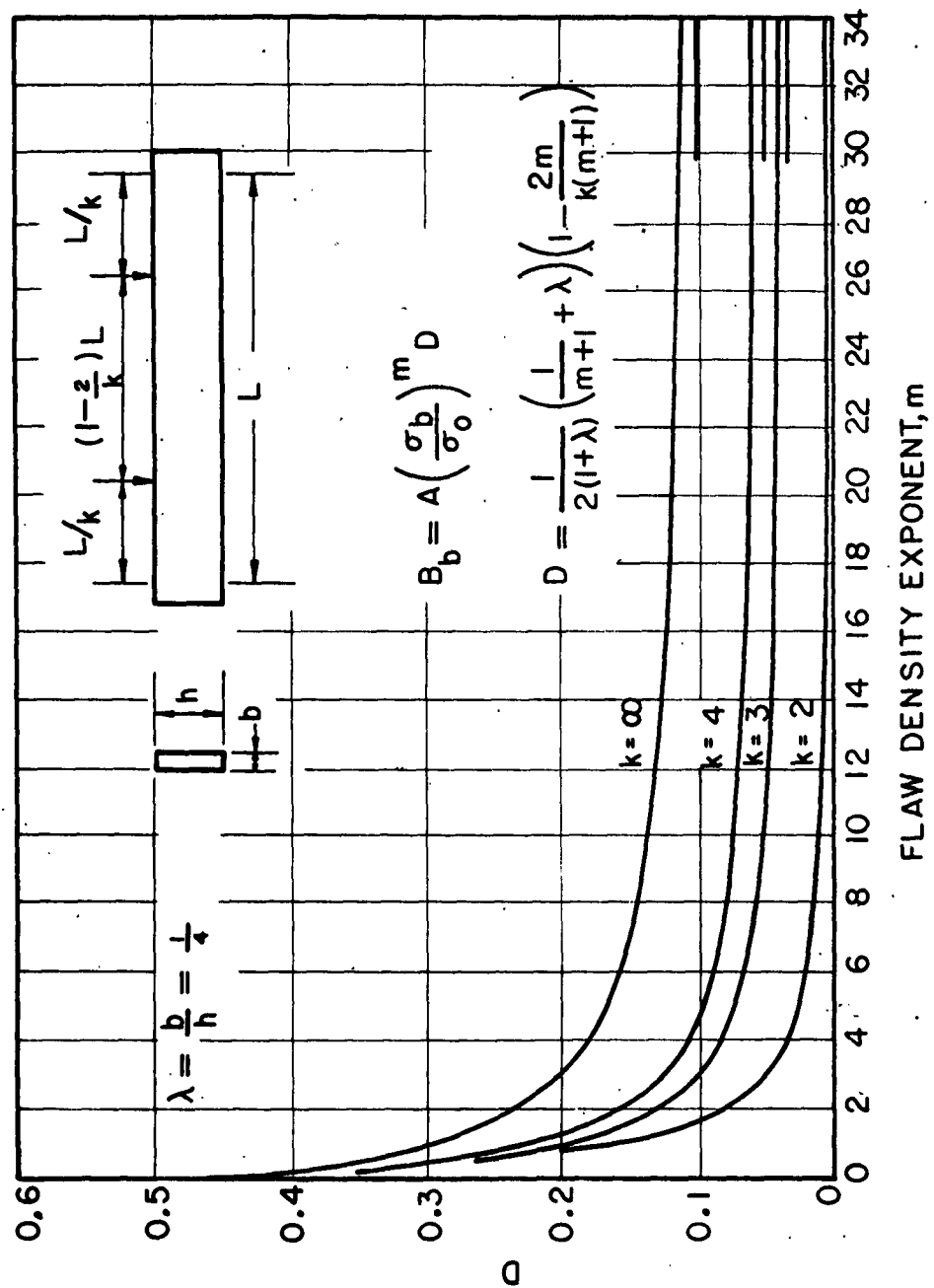$$B_t = V_t \left(\frac{\sigma_t - \sigma_u}{\sigma_o} \right)^m \tag{37}$$

- 13 -

Fig. 3 LOADING-SHAPE FACTOR D VERSUS FLAW DENSITY EXPONENT m FOR MATERIAL GOVERNED BY SURFACE FLAW DISTRIBUTION

$$\left(\frac{b}{h} = \frac{1}{4}\right)$$

$$\lambda = \frac{b}{h} = \frac{1}{4}$$

$$B_b = A\left(\frac{\sigma_b}{\sigma_o}\right)^m D$$

$$D = \frac{1}{2(1+\lambda)}\left(\frac{1}{m+1} + \lambda\right)\left(1 - \frac{2m}{k(m+1)}\right)$$
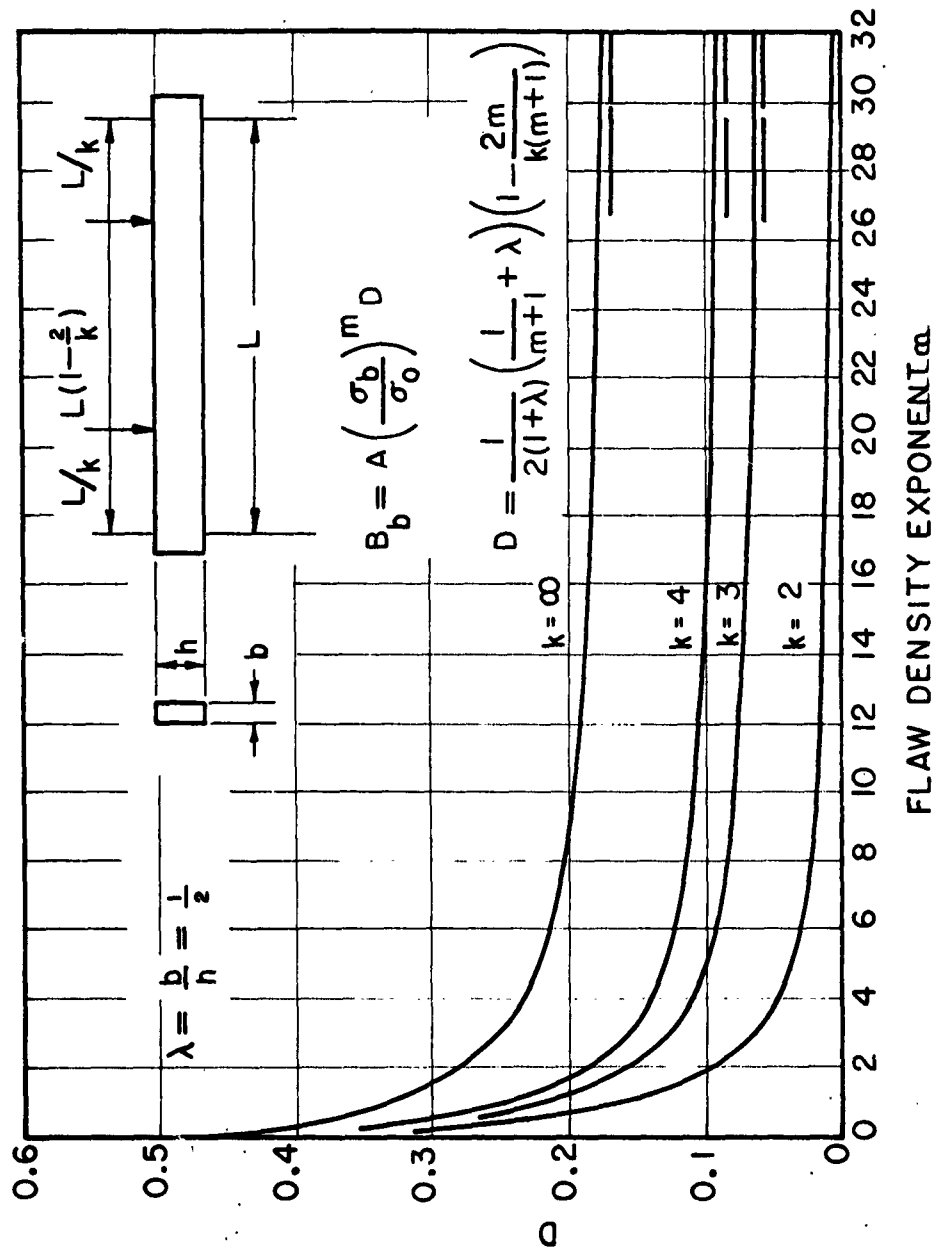
FLAW DENSITY EXPONENT, m

- 14 -

Fig. 4 LOADING-SHAPE FACTOR D VERSUS FLAW DENSITY EXPONENT m FOR MATERIAL GOVERNED BY SURFACE FLAW DISTRIBUTION $\left(\frac{b}{h} = \frac{1}{2}\right)$
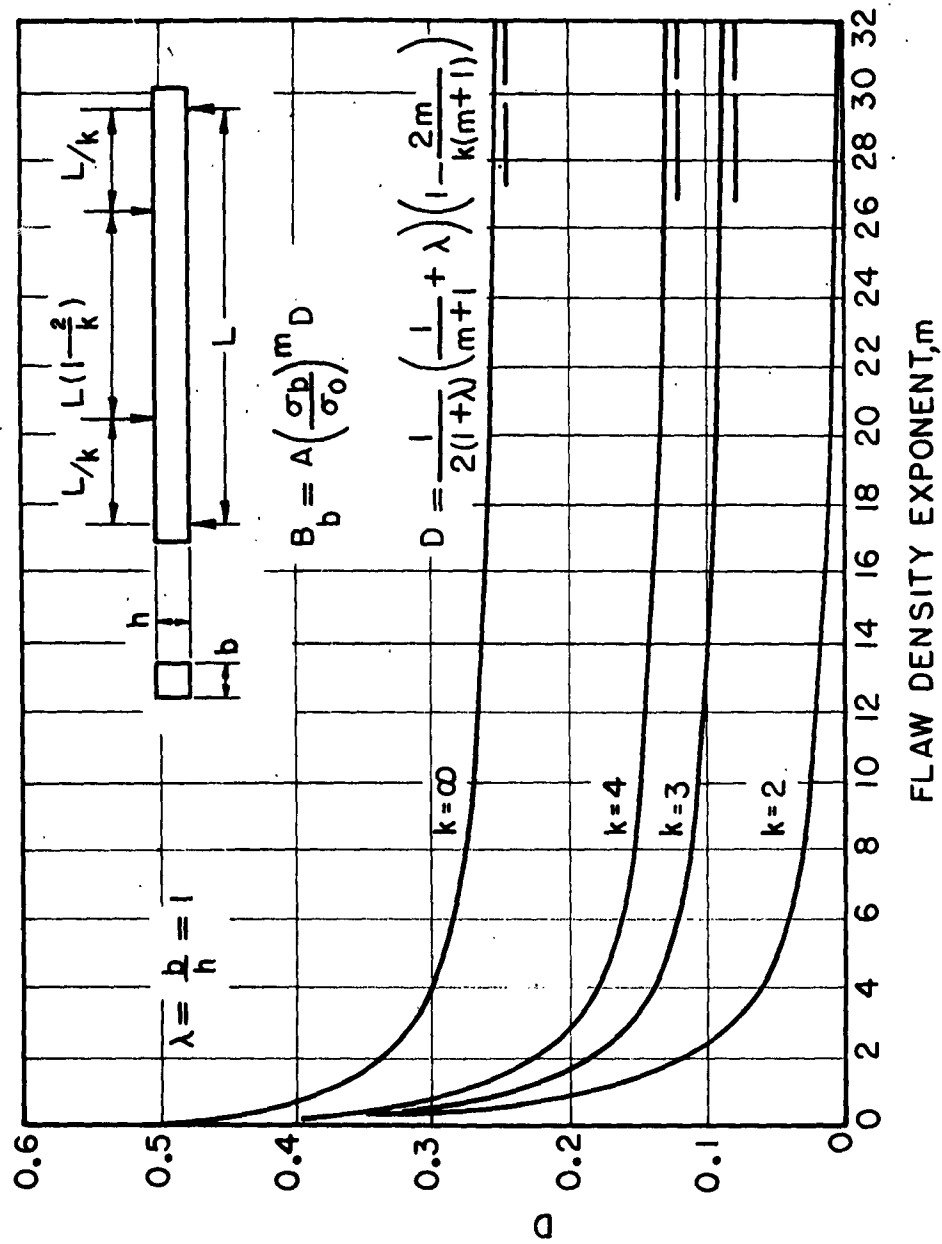
$$\lambda = \frac{b}{h} = 1$$

$$B_b = A\left(\frac{\sigma_b}{\sigma_o}\right)^m D$$

$$D = \frac{1}{2(1+\lambda)}\left(\frac{1}{m+1}+\lambda\right)\left(1-\frac{2m}{k(m+1)}\right)$$

Fig. 5   LOADING-SHAPE FACTOR D VERSUS FLAW DENSITY EXPONENT m
FOR MATERIAL GOVERNED BY SURFACE FLAW DISTRIBUTION
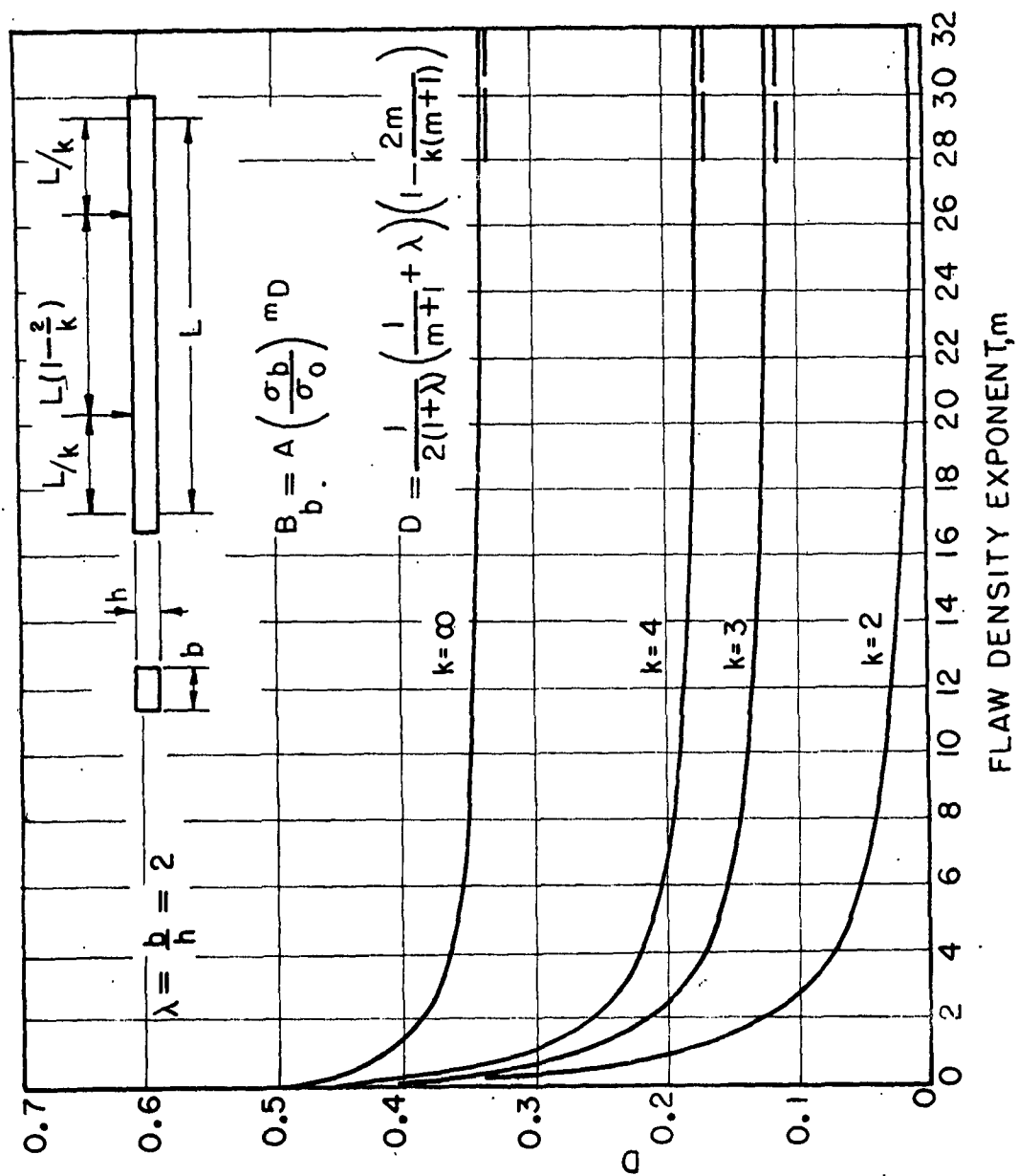$\left(\frac{b}{h} = 1\right)$

Fig. 6 LOADING-SHAPE FACTOR D VERSUS FLAW DENSITY EXPONENT m
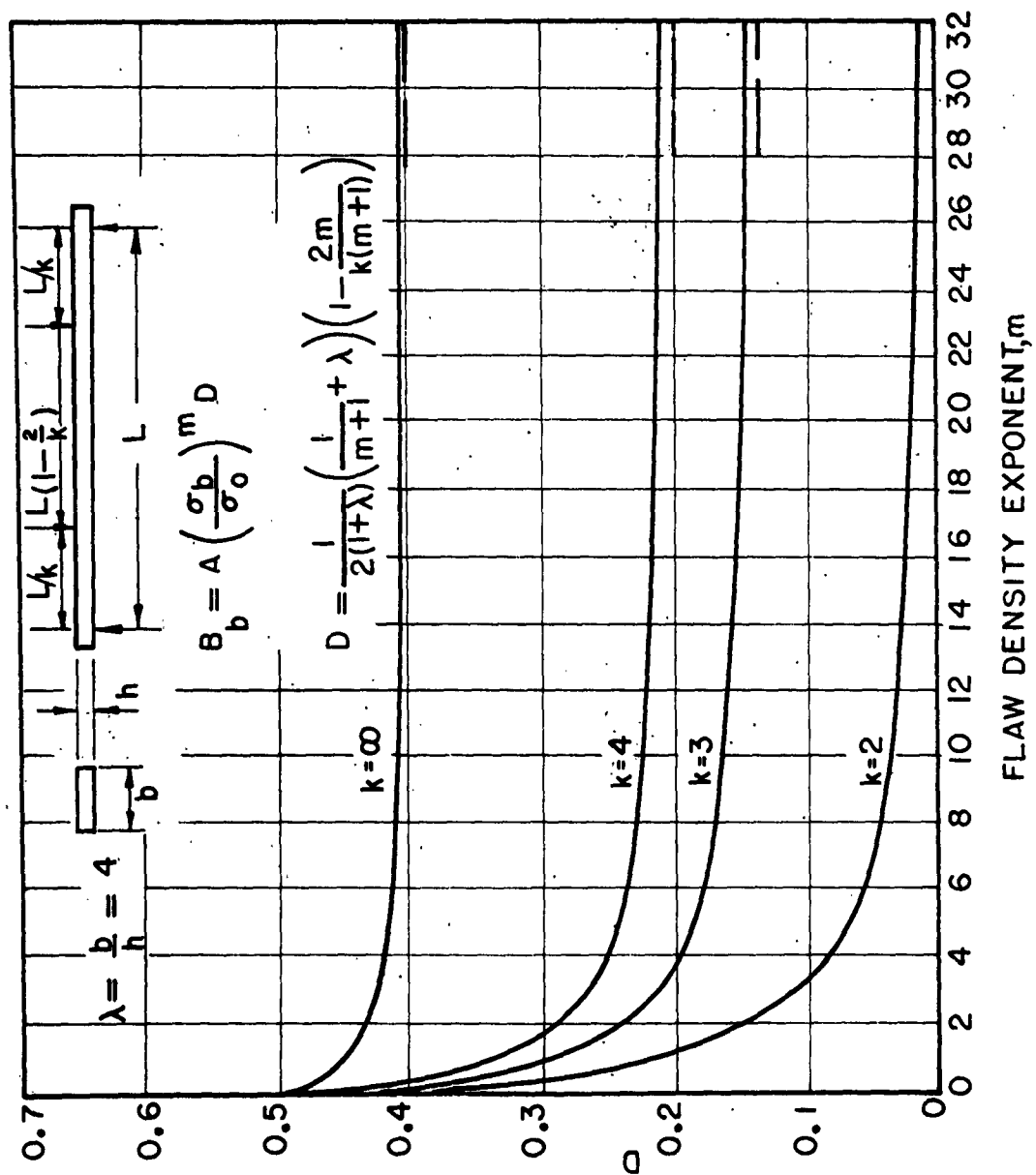FOR MATERIAL GOVERNED BY SURFACE FLAW DISTRIBUTION
$(\frac{b}{h} = 2)$

Fig. 7 LOADING-SHAPE FACTOR D VERSUS FLAW-DENSITY EXPONENT m FOR MATERIAL GOVERNED BY SURFACE FLAW DISTRIBUTION $(\frac{b}{h} = 4)$

where V_t is the volume of the specimen, whereas for a prismatic speciment under pure bending the risk of rupture was given by Eq. 14.

In order to compare mean failure stresses the risks of rupture given above must be substituted in Eq. 3 in order to compute σ_m for the two cases. However, this is quite involved for the case of pure bending, since for this case Eq. 3 cannot be solved in closed form expression. Instead, comparison of median failure stresses, or any stresses of a given probability of fracture for that matter, is easily carried out by equating instead the risks of rupture corresponding to the different loading conditions

$$V_t \left(\frac{\sigma_t - \sigma_u}{\sigma_o} \right)^m = \frac{V_b}{2(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b} \right) \left(\frac{\sigma_b - \sigma_u}{\sigma_o} \right)^m$$

from which

$$\frac{\sigma_t}{\sigma_b} = \left[\frac{1}{2(m+1)} \frac{V_b}{V_t} \right]^{1/m} \left(1 - \frac{\sigma_u}{\sigma_b} \right)^{1 + \frac{1}{m}} + \frac{\sigma_u}{\sigma_b} \tag{38}$$

For $\sigma_u = 0$ this expression reduces to

$$\frac{\sigma_t}{\sigma_b} = \left[\frac{1}{2(m+1)} \frac{V_b}{V_t} \right]^{1/m} \tag{39}$$

In this case the relationship between median stresses is the same as that between mean stresses, as can be shown easily. In the case of a classical material ($m = \infty$) Eq. 38 yields

$$\sigma_t = \sigma_b = \sigma_u$$

For a material governed by surface flaw distribution the risk of rupture in a tensile specimen of rectangular cross section (b x h) is

$$B_t = 2 L (b + h) \left(\frac{\sigma_t - \sigma_u}{\sigma_o} \right)^m \tag{40}$$

as compared to Eq. 26 which gives the risk of rupture under conditions of pure bending.

Equating risks of rupture we obtain the following relation

$$\frac{\sigma_t}{\sigma_b} = \left[\frac{(m+1) \frac{b}{h} + 1 - \frac{\sigma_u}{\sigma_b}}{2(m+1)(1 + \frac{b}{h})} \right]^{1/m} \left(1 - \frac{\sigma_u}{\sigma_b} \right) + \frac{\sigma_u}{\sigma_b} \tag{41}$$

which, for $\sigma_u = 0$, reduces to

$$\frac{\sigma_t}{\sigma_b} = \left[\frac{(m+1) \frac{b}{h} + 1}{2(m+1)(1 + \frac{b}{h})} \right]^{1/m} \tag{42}$$

V.   ANALYTICAL DETERMINATION OF MATERIAL PARAMETERS

Experimental determinations of material parameters generally call for testing a number of simple calibration specimens and passing a curve through the data points presented in a plot showing the probability of fracture against the failure stress. Given a theory, such as the Weibull theory, an expression for the probability of fracture can be obtained which is a function of specimen geometry and material parameters to achieve a "best fit". The objective is, then, to find those values of the material parameters which make the theoretical curve fit best the experimental points. A suitable criterion for this purpose is the minimization of the sum of the mean square differences. This method, developed below, is applicable to materials governed either by volumetric or surface flaw distribution.

A. Material Governed by Volumetric Flaw Distribution

The probability of fracture at the stress σ_n is

$$S_n = 1 - e^{-B_n} \tag{43}$$

Combining this with the expression for the risk of rupture given in Eq. 14, there results

$$y_n = \ln \ln \frac{1}{1-S_n} = \ln B_n$$

$$= \ln \frac{V}{2} - \ln(m+1) + (m+1)\ln(\sigma_n - \sigma_u) - \ln \sigma_n - m \ln \sigma_o \tag{44}$$

The corresponding (estimated) value of this function of probability of fracture obtained experimentally is

$$Y_n = \ln \ln \frac{N+1}{N+1-n} \tag{45}$$

where,

N is total number of specimens tested, and

n is the serial number of specimen (when specimens are ordered according to ascending values of the fracture stress σ_n).

The least squares method requires that for a best fit

$$\sum_{n=1}^{N} \left| Y_n - y_n \right|^2 = \text{minimum} \tag{46}$$

The necessary conditions for the existence of this minimum are

$$\sum_{n=1}^{N} (Y_n - y_n) \frac{\partial y_n}{\partial \sigma_u} = 0$$

$$\sum_{n=1}^{N} (Y_n - y_n) \frac{\partial y_n}{\partial \sigma_o} = 0 \qquad\qquad (47)$$

$$\sum_{n=1}^{N} (Y_n - y_n) \frac{\partial y_n}{\partial m} = 0$$

which reduce to

$$\sum_{n=1}^{N} \left[\ell n \; \ell n \; \frac{N+1}{N+1-n} - \ell n \; B_n \right] \frac{1}{\sigma_n - \sigma_u} = 0$$

$$\sum_{n=1}^{N} \left[\ell n \; \ell n \; \frac{N+1}{N+1-n} - \ell n \; B_n \right] = 0 \qquad\qquad (48)$$

$$\sum_{n=1}^{N} \left[\ell n \; \ell n \; \frac{N+1}{N+1-n} - \ell n \; B_n \right] \ell n \left(\frac{\sigma_n - \sigma_u}{\sigma_o} \right) = 0$$

B.   Material Governed by Surface Flaw Distribution

Using in this case the expression for the risk of rupture given by Eq. 26, the theoretical relation between the probability of fracture, S_n, and the corresponding failure stress, σ_n, becomes

$$y_n = \ell n \; \ell n \; \frac{1}{1 - S_n} = \ell n \; B_n =$$

$$= \ell n \; L + m \ell n \, (\sigma_n - \sigma_u) - m \ell n \; \sigma_o + \ell n \left[\frac{h}{(m+1)} \left(1 - \frac{\sigma_u}{\sigma_n} \right) + b \right] \qquad (49)$$

Application of the least squares method again leads to Eqs. 47, which in the present case reduce to

$$\sum_{n=1}^{N} \left[\ell n \ \ell n \ \frac{N+1}{N+1-n} \ - \ \ell n \ B_n \right] \left[\frac{m}{\sigma_n - \sigma_u} \ + \ \frac{h}{\left[\frac{h}{m+1} \left(1 - \frac{\sigma_u}{\sigma_n} \right) + b \right] (m+1) \ \sigma_n} \right] = 0$$

$$\sum_{n=1}^{N} \left[\ell n \ \ell n \ \frac{N+1}{N+1-n} \ - \ \ell n \ B_n \right] = 0$$

$$\sum_{n=1}^{N} \left[\ell n \ \ell n \ \frac{N+1}{N+1-n} \ - \ \ell n \ B_n \right] \left[\frac{h \left(1 - \frac{\sigma_u}{\sigma_n} \right)}{\left[\frac{h}{m+1} \left(1 - \frac{\sigma_u}{\sigma_n} \right) + b \right] (m+1)^2} \ - \right.$$

$$\left. - \ \ell n \ \left(\frac{\sigma_n - \sigma_u}{\sigma_o} \right) \right] = 0 \tag{50}$$

Equations 48 and 50, depending on the type of material, are solved for the three material parameters σ_u, σ_o, and m. They are too involved for a conventional solution to be attempted and a computer solution is required.

VI.   EXPERIMENTAL WORK

The material used for the experimental evaluation of the method described above was Columbia Resin (CR-39), an amorphous, brittle polymer. Thirty-six CR-39 specimens 0.4 in. x 0.4 in. x 4 in. were cut from a 1/2 in. thick sheet with all surfaces of each specimen having a uniformly machined finish. These specimens were tested under four-point loading with a gage length of 2 in. subjected to pure bending and a distance of 3/4 in. between

loads and supports. The existence of pure bending between the two middle
loads is illustrated by the isochromatic fringe pattern of Fig. 8, obtained
on one such specimen under four-point loading. The specimens were tested
in an Instron model TT testing machine with a crosshead speed of 0.5 in. /min.
A load versus crosshead deflection record was obtained in each case, and
deviation from linearity was noted. Although this fact does not preclude
localized non-linear deformation, the data were analyzed on the basis of
linear elastic behavior to failure. The following results were obtained:

Mean failure stress   : σ_m = 6,460 psi
Standard deviation   : a = 1,060 psi
Variance   : a^2 = 1,123,600 $(psi)^2$
Coefficient of variation : v = 16.41%
Highest failure stress : σ_{high} = 8,780 psi
Lowest failure stress   : σ_{low} = 3,670 psi

A volumetric flaw distribution was assumed and Eqs. 48 were
solved by the computer for the desired material parameters. The
following values were obtained:

σ_u = 940 psi
σ_o = 3030 psi
m = 5.79

The theoretical cumulative distribution curve of Eq. 43 for the values of the
parameters found above was plotted in Fig. 9 along with the experimental
points based on the relation

$$S_n = \frac{n}{N+1}$$

It is seen that the "fit" is very satisfactory. It should be remarked, however,
that the nature of the fitting criterion used (minimization of least squares)
does not necessarily guarantee a best fit in other respects; e.g. the
derivative of the cumulative distribution function which provides the
probability density function, may not show an equally satisfactory correlation.
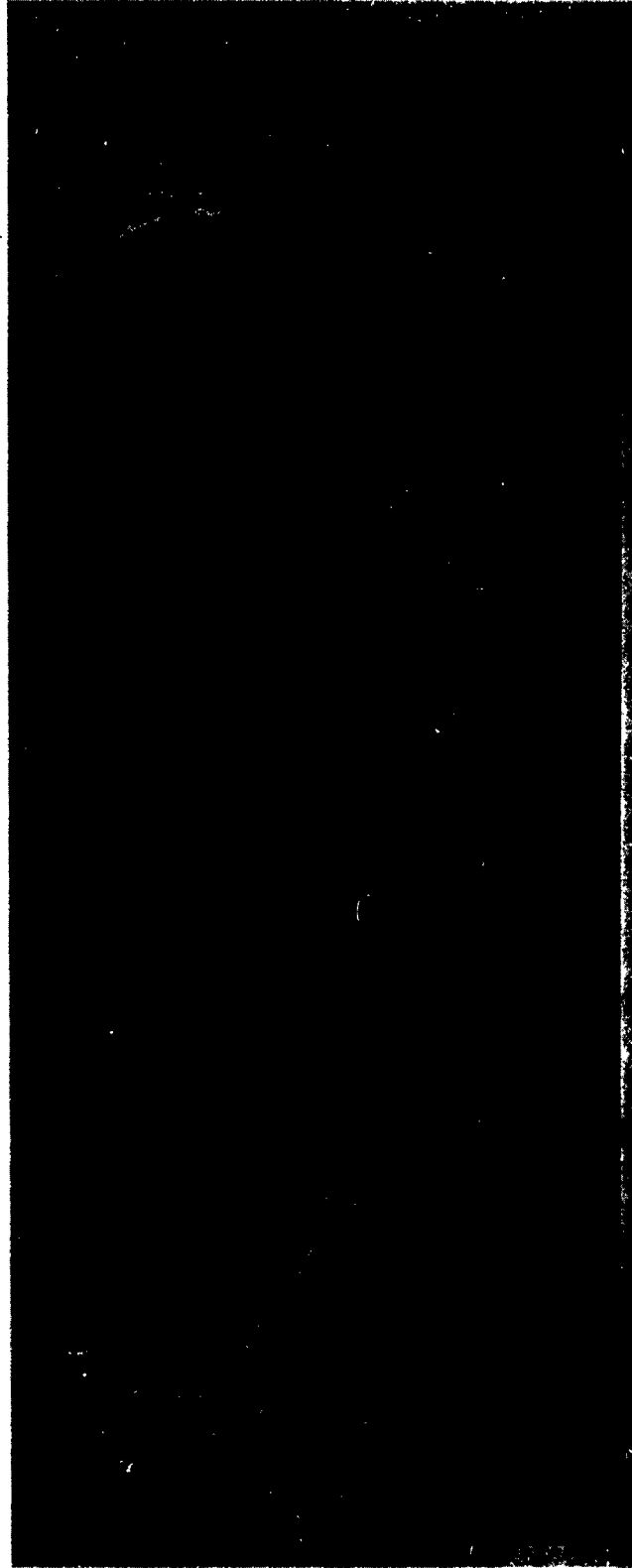
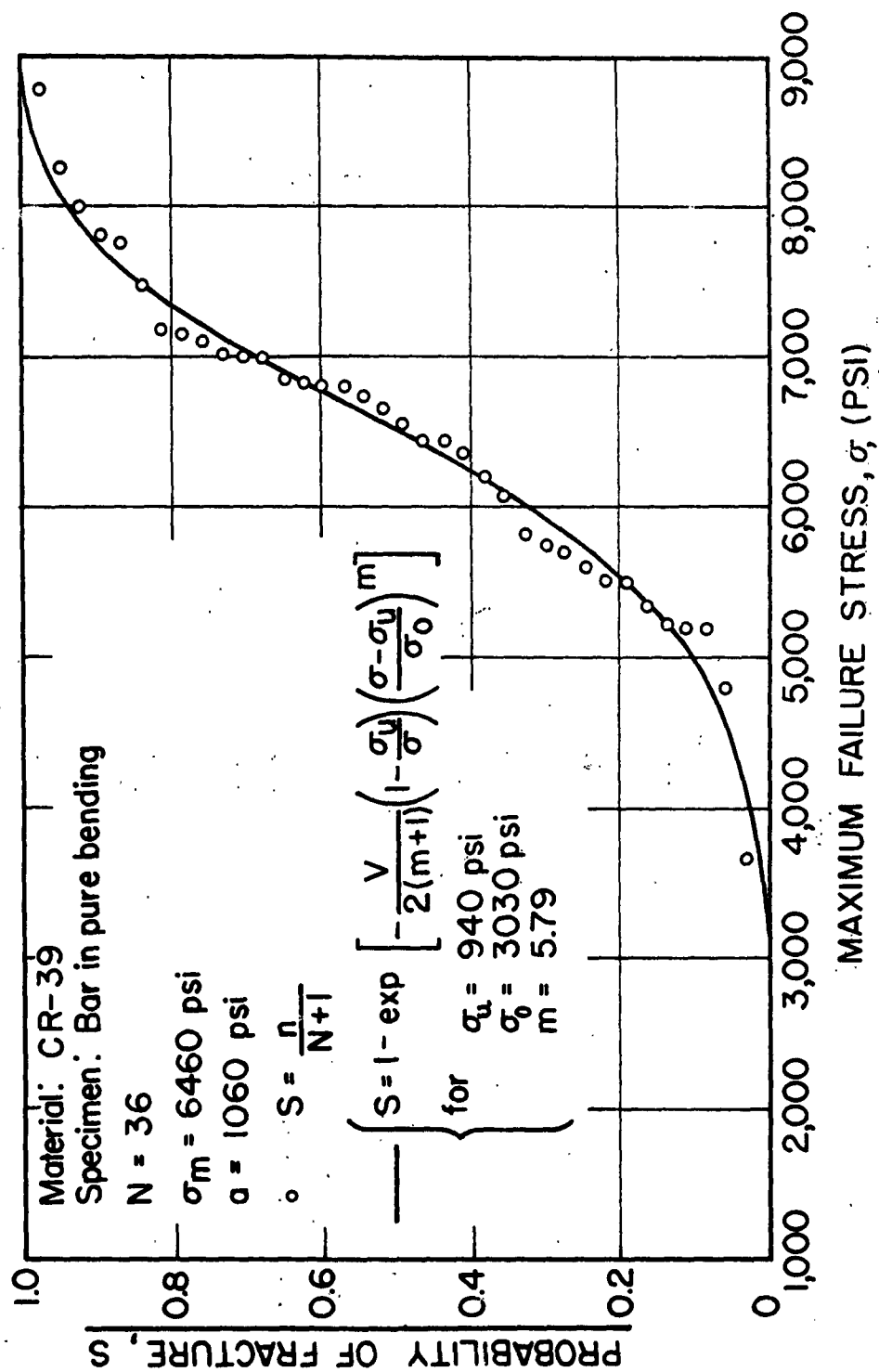Fig. 8 ISOCHROMATIC FRINGE PATTERN IN CR-39 SPECIMEN UNDER
FOUR-POINT LOADING

Fig. 9 COMPARISON OF EXPERIMENTAL VALUES WITH THEORETICAL CUMULATIVE

DISTRIBUTION CURVE OF BEST FIT

VII. DISCUSSION

Equation 12 shows that, for materials whose fracture is governed by a volumetric flaw distribution, the only specimen dimension entering the expression for the risk of rupture is the total volume V. For a fixed value of the parameter k variations of length, width or depth of specimen which leave its volume unaffected will not affect the risk of rupture and, therefore, the latter is independent of (transversal) stress gradient.

The dependence of the risk of rupture on specimen dimensions is more complicated for the case of a material governed by a surface flaw distribution. For a given material σ_o, σ_u and m are constants. If one is interested solely in the effect of depth and width upon the risk of rupture, h and b should be regarded as being the only variables, while the length L and extreme fiber stress σ_b should be kept constant. Then, the risk of rupture for a pure bending specimen, as given by Eq. 26, remains constant provided b and L satisfy the relationship

$$\frac{h}{m+1} \left(1 - \frac{\sigma_u}{\sigma_b} \right) + b = \text{constant}$$

Substituting now $h = \dfrac{2\sigma_b}{g}$, where g is the stress gradient, there results

$$\frac{2\sigma_b}{g(m+1)} \left(1 - \frac{\sigma_u}{\sigma_b} \right) + b = \text{constant, or}$$

$$\frac{C_1}{g} + b = C \tag{51}$$

where

$$C_1 = \frac{2}{m+1} \ (\sigma_b - \sigma_u)$$

$$C = \text{constant}$$

From Eq. 51 it follows that it is possible to vary the stress gradient without

affecting the risk of rupture, provided the width b varies in such a manner that Eq. 51 is satisfied, with appropriate reference to the strength of the material, σ_b.

For purposes of material parameter determination any type of bending test would be suitable if $\sigma_u = 0$. If $\sigma_u \neq 0$ material parameters can be uniquely determined from a tensile or a pure bending test. This means that for four-point loading the loads should be as close to the supports as practicable, yet without incurring the risk of causing shear failures to develop next to the support points. In general, it is possible to apply the equations for pure bending to results obtained from tests of beams subjected to symmetric four-point loading if only fractures occurring in the portion of the beam under constant bending moment are regarded as being valid. Tests resulting in fracture outside the gage length of interest(between supports and loading points) would be discarded from the statistical analysis.

The conventional trial and error method for the determination of parameters is suitable for tensile, torsion and pure bending specimens in the case of volume-distributed flaws and only for tensile and torsion specimens in the case of surface distributed flaws. The method as it is usually applied is not entirely free of subjectivity. The analytical method described here is more general and is especially advantageous in the case of surface-distributed flaws. Its successful application was clearly demonstrated in the case of CR-39.

VIII. CONCLUSION

Statistical failure theories developed to date implicitly assume that the probability of fracture is governed by the tensile normal stresses existing in an element, and is independent of the intensity of shear stresses existing in any part of the body. The overall fracture probability of a piece is then obtained by summing up the individual failure probabilities of infinitesimal components composing the body, this result remaining independent of the stress gradient accompanying the internal stress distribution.

The validity of this hypothesis has never been examined. This investigation, therefore, was aimed at evolving a carefully conceived approach for investigating whether the existence of stress gradients had an independent influence upon the fracture characteristics of brittle ceramic substances.

This paper presents the results of the theoretical part of the study. The Weibull theory was adopted as the basis of characterization of the probabilistic fracture behavior of brittle substances. The symmetrically loaded beam under four-point bending was selected as the principal subject of analysis, both because it represents one of the easiest shapes for analysis and experiments, and because it is one of the geometrical configurations that can yield conclusive proof regarding the influence of stress gradients through the simple expedient of changing the proportions but retaining the area of the rectangular cross section.

A theoretical treatment was developed for the completely general case of an arbitrary positioning of the two symmetrically disposed loads on the beam, for the case of a non-integral value of m and a non-vanishing value of σ_u. Solutions of this nature were obtained for both of the broad cases possible with the Weibull theory, that is the case significant flaws uniformly dispersed either volumetrically or confined to the surface. Substantial simplifications in the resulting expressions are shown to be possible first if m is an integer and, second, if σ_u has a vanishing value. The special cases of pure bending, quadrant-point, third-point and center-point loading were derived, the first and the last of these representing the limiting cases of the general solution.

Because no completely satisfactory treatment of a most reliable fit of the Weibull probability density curve to experimental data exists in the current literature (for $\sigma_u \neq 0$) an analytical method for obtaining the "best fit" of a theoretical curve to a set of test points is presented, following the analytical derivations for the effect of non-uniform stresses. The process producing this best fit is based upon a minimization of the sum of the mean squares of theoretical and experimental data, which then yield the most reliable balues of three parameters descriptive of the Weibull distribution. Lastly, the methodology so developed is illustrated by applying it

to a set of test data obtained with specimens of CR-39, a brittle polymeric material.

Experimental work in support of the analytical studies presented here was recently completed and presented in another paper [1].

IX. ACKNOWLEDGEMENT

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